Nebraska Wesleyan University | Test 4B
Math 060 | Spring 2014
Instructor: Austin Mohr

Please write only your name on this sheet.
Write all work and answers on the blank sheets.
Only attempt problems for which you have not already received a grade of "Master".
Use algebra or calculus techniques to solve these problems. Appealing to a graph will not earn any credit.

13. We will make the following assumptions about the sales of a particular commodity.

- Supply is given by $S(x) = 7000 + 5x$, where $S(x)$ is the price (in dollars) the supplier will charge when x items are produced.
- Demand is given by $D(x) = 8800 - 3x$, where $D(x)$ is the price (in dollars) consumers will pay when x items are produced.
- Price and production correspond to the equilibrium point of supply and demand.

Under all these assumptions, determine the producer surplus.

Solution: First we must determine the equilibrium point by solving

$$\begin{aligned} 7000 + 5x &= 8800 - 3x \\ 8x &= 1800 \\ x &= 225. \end{aligned}$$

At this level of production, the price will be set at $S(225) = 7000 + 5 \cdot 225 = \$8,125$ dollars.

The producer surplus is the value received by the producer above its actual cost, which is given by

$$\begin{aligned} \int_0^{225} 8125 - S(x)\, dx &= \int_0^{225} 1125 - 5x\, dx \\ &= \left[1125x - \frac{5}{2}x^2\right]_0^{225} \\ &= \left(1125 \cdot 225 - \frac{5}{2} \cdot 225^2\right) - \left(1125 \cdot 0 - \frac{5}{2} \cdot 0^2\right) \\ &\approx \$126,562.50. \end{aligned}$$

14. Calculate the value of k required in order for each function to be used as a probability density function.

(a) kx^2 on the domain $[-2, 2]$

(b) ke^{-5x} on the domain $[0, \infty)$

Solution: For part a, we solve

$$\int_{-2}^{2} kx^2\,dx = k\int_{-2}^{2} x^2\,dx$$

$$k\left[\frac{x^3}{3}\right]_{-2}^{2}$$

$$\begin{aligned}
&= \frac{k}{3}\left[x^3\right]_{-2}^{2} \\
&= \frac{k}{3}\left(2^3-(-2)^3\right) \\
&= \frac{16k}{3}.
\end{aligned}$$

Since we require $\frac{16k}{3}=1$ for a probability density function, we must choose $k=\frac{3}{16}$.

For part b, we solve

$$\begin{aligned}
\int_0^\infty ke^{-5x}\,dx &= k\int_0^\infty e^{-5x}\,dx \\
&= k\left[-\frac{1}{5}e^{-5x}\right]_0^\infty \\
&= -\frac{k}{5}\left[e^{-5x}\right]_0^\infty.
\end{aligned}$$

For ease of reading, let's compute $\left[e^{-5x}\right]_0^\infty$ separately:

$$\begin{aligned}
\left[e^{-5x}\right]_0^\infty &= \lim_{b\to\infty} e^{-5b} - e^{-5\cdot 0} \\
&= \lim_{b\to\infty} \frac{1}{e^{5b}} - 1 \\
&= 0-1 \\
&= -1.
\end{aligned}$$

Returning to where we left off,

$$-\frac{k}{5}\left[e^{-5x}\right]_0^\infty = -\frac{k}{5}\cdot(-1)$$

$$\frac{k}{5}.$$

Since we require $\frac{k}{5}=1$ for a probability density function, we must choose $k=5$.

15. A random variable X takes on values in the domain $[1,\infty)$ and is distributed according to the probability density function $\frac{3}{x^4}$.

    (a) Calculate $P(X \le 2)$.

(b) Calculate the expected value of X.

Solution: For part a, we use the definition of probability density function to compute

$$\begin{aligned} P(X \leq 2) &= \int_1^2 \frac{3}{x^4}\,dx \\ &= 3\int_1^2 x^{-4}\,dx \\ &= 3\left[\frac{x^{-3}}{-3}\right]_1^2 \\ &= -\left[\frac{1}{x^3}\right]_1^2 \\ &= -\left(\frac{1}{2^3} - \frac{1}{1^3}\right) \\ &= \frac{7}{8} \\ &= 87.5\%. \end{aligned}$$

For part b, we use the definition of expected value to compute

$$\begin{aligned} E(X) &= \int_1^\infty x\frac{3}{x^4}\,dx \\ &= 3\int_1^\infty x^{-3}\,dx \\ &= 3\left[\frac{x^{-2}}{-2}\right]_1^\infty \\ &= -\frac{3}{2}\left[\frac{1}{x^2}\right]_1^\infty. \end{aligned}$$

For ease of reading, let's compute $\left[\frac{1}{x^2}\right]_1^\infty$ separately.

$$\begin{aligned} \left[\frac{1}{x^2}\right]_1^\infty &= \lim_{b\to\infty}\frac{1}{b^2} - \frac{1}{1^2} \\ &= 0 - 1 \\ &= -1. \end{aligned}$$

Returning to where we left off,

$$\begin{aligned} -\frac{3}{2}\left[\frac{1}{x^2}\right]_1^\infty &= -\frac{3}{2}\cdot(-1) \\ &= \frac{3}{2}. \end{aligned}$$

16. Scores on a standardized test are normally distributed with mean 75 and standard deviation 10.

    (a) Calculate the probability that a randomly chosen student scores between 65 and 80.

    (b) Calculate the probability that a randomly chosen student scores lower than 65.

Solution: Let X be the normal random variable corresponding to the test scores and let Z be its standard normal counterpart.

For part a, we need

$$\begin{aligned} P(60 \leq X \leq 80) &= P\left(\frac{65-75}{10} \leq \frac{X-75}{10} \leq \frac{80-75}{10}\right) \\ &= P\left(-1.0 \leq Z \leq 0.5\right) \\ &= P\left(0 \leq Z \leq 1\right) + P\left(0 \leq Z \leq 0.5\right) \\ &= 0.3413 + 0.1915 \\ &= 0.5328. \end{aligned}$$

For part b, we need

$$\begin{aligned} P(X \leq 50) &= P\left(\frac{X-75}{10} \leq \frac{65-75}{10}\right) \\ &= P\left(Z \leq -1.0\right) \\ &= 0.5 - P\left(-1.0 \leq Z \leq 0\right) \\ &= 0.5 - P\left(0 \leq Z \leq 1.0\right) \\ &= 0.5 - 0.3413 \\ &= 0.1587. \end{aligned}$$