Nebraska Wesleyan University	Test 2
Math 060	Solutions
Instructor: Austin Mohr	Spring 2014

Please write only your name on this sheet.
Write all work and answers on the blank sheets.
Only attempt problems for which you have not already received a grade of "Master".

5. Answer each of the following questions about the function $f(x) = \frac{1}{3}x^3 - x$.

   (a) On what intervals is the function concave up? On what intervals is it concave down?

   (b) On what intervals is the function increasing? On what intervals is it decreasing?

   (c) What are the *coordinates* of the local extrema (if any)?

   (d) What are the *coordinates* inflection points (if any)?

   Solution: The questions are not asked in a natural order, so we will find various answers as we proceed.

   The first derivative pertains to rate of change and extrema. Calculate $f'(x) = x^2 - 1$, which is equal to 0 when $x = -1$ and $x = 1$. We can test points in the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

x	$f'(x)$	Conclusion
-2	1	Increasing on $(-\infty, -1)$
0	-1	Decreasing on $(-1, 1)$
2	1	Increasing on $(1, \infty)$

   This data shows us there is a local maximum at $x = -1$ and a local minimum at $x = 1$. Using these values in the original function tells us that the coordinates of the local maximum are $(-1, \frac{2}{3})$ and the coordinates of the local minimum are $(1, -\frac{2}{3})$.

   The second derivative pertains to concavity and inflection points. Calculate $f''(x) = 2x$, which is equal to 0 when $x = 0$. We can test points in the intervals $(-\infty, 0)$ and $(0, \infty)$.

x	$f'(x)$	Conclusion
-1	-2	Concave down on $(-\infty, 0)$
1	2	Concave up on $(0, \infty)$

   This data shows us there is an inflection point at $x = 0$. Using this value in the original function tells us that the coordinates of the inflection point are $(0, 0)$.

6. Provide the equation of the requested asymptote for each function.

   (a) $\frac{x+1}{x^2-2x-3}$, vertical

   (b) $\frac{3x^2+4x+10}{x^3}$, horizontal

   (c) $\frac{x^2+2x-1}{x-3}$, slant (Hint: $\frac{x^2+2x-1}{x-3} = x + 5 + \frac{14}{x-3}$)

   Solution: In part (a), the vertical asymptote is given when the denominator equals 0. Factoring shows that this occurs when $x = -1$ and $x = 3$.

   In part (b), the horizontal asymptote is $y = 0$, since the degree of the denominator is greater than the degree of the numerator.

   In part (c), the slant asymptote is $y = x + 5$, since the term $\frac{14}{x-3}$ is negligible when x is large.

7. Find the *coordinates* of the global maximum and global minimum (if they exist) for the function $f(x) = 3x^2 - 2x$ on the specified interval. If one of the global extrema does not exist, explain why it does not exist.

   (a) On the interval $[0, 1]$

(b) On the interval $[1, \infty)$

Solution: In part (a), we have a continuous function on a closed interval, so the Extreme Value Theorem guarantees absolute extrema at critical values or at endpoins of the interval. We first compute the derivative $f'(x) = 6x - 2$, which is equal to 0 when $x = \frac{1}{3}$. We check this critical point together with the endpoints of the domain.

x	$f(x)$
0	0
$\frac{1}{3}$	$-\frac{1}{3}$
1	1

Therefore, the absolute maximum on this domain has coordinates $(1, 1)$ and the absolute minimum has coordinates $(\frac{1}{3}, -\frac{1}{3})$.

In part (b), we have an unbounded domain, so the Extreme Value Theorem does not apply. We already know that the only critical value of $f(x)$ is $x = \frac{1}{3}$, which is outside our domain. Checking a value to the right of this, such as $x = 2$, in the first derivative shows that the function is increasing on the entire interval $[1, \infty)$. Therefore, the global minimum is the point $(1, 1)$ and there is no global maximum.

8. A hotel can fill all 400 of its rooms if it charges only \$90 per day. For every increase of x dollars in the daily room rate, there are x rooms vacant. Each occupied room costs \$30 per day to service and maintain. What should the hotel charge per day for a room in order to maximize profit? What is the maximum profit?

   Solution: The profit earned by the hotel is given by the number of rooms filled multiplied by the profit per room. Each of these are functions of x, the number of dollars beyond \$90 that we charge for the room. The number of rooms filled is $400 - x$, since we lose a room for each dollar increase. The profit per room is $90 + x - 30$ (or just $60 + x$), since we charge $90 + x$ dollars for the room but lose \$30 to maintain the room. The total profit as a function of x is given by $P(x) = (400 - x)(60 + x)$, which can be written more usefully as $P(x) = 24000 + 340x - x^2$. To optimize the function, we compute the derivative $P'(x) = 340 - 2x$, which is equal to 0 when $x = 170$. This means the hotel should charge \$260, for a total profit of \$52,900.